Boundary integral method for quantum billiards in a constant magnetic field

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We derive a boundary integral equation to compute the eigenvalues of two-dimensional billiards subjected to a magnetic field. The integral requires the Green's function of the boundary-free problem with the magnetic field pointing in the opposite direction. This Green's function is computed for the case of a constant magnetic field perpendicular to the billiard and some applications are discussed. The elliptical billiard is then studied numerically as an example of a nontrivial application. [S1063-651X(96)05612-7]

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I. INTRODUCTION

Boundary integral methods constitute a powerful tool in the solution of Helmholtz's equation $(\nabla^2+k^2)\psi=0$ with given boundary conditions for ψ. The history and a survey of these techniques can be traced from Refs. [1,2]. One of the most important applications of this method is in the quantum mechanics of chaotic billiards, where it has been largely used to compute eigenvalues of Schrödinger's equation [3]. The definition of a *quantum surface of section* by Bogomolny [5] and its application to a one-dimensional Sturm-Liouville problem [6] are examples of the versatility of boundary methods to solve nontrivial quantum-mechanical eigenvalue problems. The idea of these methods is to obtain a matrix, depending only on the shape of the boundary, whose determinant has zeros at the right eigenvalues.

In the past few years there has been an increasing interest in the behavior of confined particles subjected to a uniform magnetic field [7–14]. Billiards have been used successfully in several situations to model such confining potentials. In this case, Schrödinger's equation reads $(m=e=c=1)$

$$\frac{1}{2}(-i\hbar\boldsymbol{\nabla}-\mathbf{A})^2\psi=E\psi, \tag{1}$$

with $\mathbf{B}=\boldsymbol{\nabla}\times\mathbf{A}$, and it cannot be reduced to Helmholtz's equation.

The computation of the eigenvalues and eigenfunctions are more elaborate and only in very simple situations it can be performed by a direct diagonalization of the Hamiltonian operator. This is because an appropriate set of basis states for such diagonalization should satisfy the billiard boundary conditions. If the billiard is integrable at zero magnetic field and its wave functions can be computed analytically (for example, square and circular billiards) the diagonalization scheme with these wave functions as a basis works fairly well [15,16]. In a general case such bases sets are not available beforehand and have to be obtained numerically. This introduces errors and makes the whole process very expensive computationally.

In a recent paper [15], a method for computing the eigenvalues and eigenfunctions of billiards in a constant magnetic field was developed. The basic idea of this method is to write the wave function as a linear combination of the boundary-free solutions, including those that diverge at infinity. The solution of the billiard problem is then obtained by imposing that the correct combination goes to zero at the boundary. Although efficient, this procedure requires an expansion of the wave function valid in the whole space, not only at the boundary.

The main result of this paper is a method for the computation of eigenvalues of billiards in a magnetic field. Using Green's identity, we are capable of generalizing the usual boundary integral determinant obtained from Helmholtz's equation. The free field case makes use of the Green's function

$$G(\mathbf{r},\mathbf{r}';E)=\frac{2\pi i}{\hbar^2}H_0^{(1)}(k|\mathbf{r}-\mathbf{r}'|). \tag{2}$$

The Hamiltonian for a particle subject to a constant magnetic field may be cast into a harmonic-oscillator problem [17], allowing us to obtain the corresponding Green's function. This is done in Sec. II.

This paper is organized as follows. In Sec. II we derive the boundary integral equation whose solution gives implicitly the eigenvalues of a generic billiard in a magnetic field. In Sec. III we derive the Green's function for the case of a constant magnetic field perpendicular to the billiard and in Sec. IV we derive an alternative quantization condition for smooth boundaries using the Fourier expansion of the Green's function. We then apply our method to compute numerically the eigenvalues of the ellipse billiard, which is nonintegrable for nonzero magnetic field.

II. THE BOUNDARY INTEGRAL EQUATION

The Schrödinger equation for a particle moving inside a billiard of domain $\mathcal{B}$ and boundary $\partial\mathcal{B}$ subjected to a magnetic field $\mathbf{B}$ can be written in the form

$$\left[\nabla^2-2\frac{i}{\hbar}\mathbf{A}\cdot\boldsymbol{\nabla}+k^2-A^2/\hbar^2\right]\psi(\mathbf{r})=0, \tag{3}$$

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where $k^2 = 2E/\hbar^2$ and the Coulomb gauge ($\boldsymbol{\nabla}\cdot\mathbf{A}=0$) is assumed. The wave function ψ should satisfy the boundary condition $\psi(\mathbf{r})=0$ for $\mathbf{r}$ on $\partial\mathcal{B}$.

We now consider the equation for the Green's function $\overline{G}(\mathbf{r},\mathbf{r}_0)$ of a particle subjected to the magnetic field $-\mathbf{B}$:

$$\left[\nabla^2 + 2\frac{i}{\hbar}\mathbf{A}\cdot\boldsymbol{\nabla} + k^2 - A^2/\hbar^2\right]\overline{G}(\mathbf{r},\mathbf{r}_0) = -\frac{8\pi}{\hbar^2}\,\delta(\mathbf{r}-\mathbf{r}_0). \tag{4}$$

Multiplying Eq. (4) on the left by ψ, subtracting the result from Eq. (3) multiplied on the left by $\overline{G}$, and integrating over the billiard area $\mathcal{B}$ we get

$$\int_{\mathcal{B}}(\overline{G}\nabla^2\psi - \psi\nabla^2\overline{G})d^2r - \frac{2i}{\hbar}\int_{\mathcal{B}}(\overline{G}\mathbf{A}\cdot\boldsymbol{\nabla}\psi + \psi\mathbf{A}\cdot\boldsymbol{\nabla}\overline{G})d^2r$$
$$= \frac{8\pi}{\hbar^2}\,\psi(\mathbf{r}_0). \tag{5}$$

The first integral can be reduced to an integral over the billiard boundary by using Green's identity as usual. The second integral can be rewritten as

$$\frac{2i}{\hbar}\int_{\mathcal{B}}(\overline{G}\mathbf{A}\cdot\boldsymbol{\nabla}\psi + \psi\mathbf{A}\cdot\boldsymbol{\nabla}\overline{G})d^2r = \frac{2i}{\hbar}\int_{\mathcal{B}}\mathbf{A}\cdot(\boldsymbol{\nabla}\overline{G}\psi)d^2r$$
$$= \frac{2i}{\hbar}\int_{\mathcal{B}}\boldsymbol{\nabla}\cdot(\mathbf{A}\overline{G}\psi)d^2r,$$

where we have used that $\boldsymbol{\nabla}\cdot\mathbf{A}=0$. We now use the Stoke theorem in the form

$$\int_{\mathcal{B}}\left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y}\right)dx\,dy = \int_{\partial\mathcal{B}}(L\,dx + M\,dy),$$

with $M = A_x\overline{G}\psi$ and $L = -A_y\overline{G}\psi$. This completely reduces Eq. (5) to a boundary integral equation

$$\int_{\partial\mathcal{B}}(\overline{G}\boldsymbol{\nabla}\psi - \psi\boldsymbol{\nabla}\overline{G})\cdot\hat{\mathbf{n}}\,dl - \frac{2i}{\hbar}\int_{\partial\mathcal{B}}\overline{G}\psi(\mathbf{A}\cdot\hat{\mathbf{n}}\,dl) = \frac{8\pi}{\hbar^2}\,\psi(\mathbf{r}_0), \tag{6}$$

where $\hat{\mathbf{n}}\,dl$ is the normal differential along the boundary.

For billiards with hard walls ψ vanishes at the boundary. In this case, we choose $\mathbf{r}_0$ at the boundary and impose that the Green's function satisfy boundary conditions *other* than those of ψ. With these choices Eq. (6) yields

$$\int_{\partial\mathcal{B}}\overline{G}\boldsymbol{\nabla}\psi\cdot\hat{\mathbf{n}}\,dl = 0. \tag{7}$$

This equation is identical in form to that obtained for the field-free case [4]. The only difference is that here the Green's function satisfies Eq. (4) and is not just a Hankel function like in the zero-field case. We shall compute this Green's function in Sec. III.

Equation (7) can be easily transformed into a determinant whose zeros give the correct eigenvalues of the billiard. To this end, we parametrize the boundary by a continuous variable l running from 0 to L, the billiard perimeter, and we write the normal derivative of ψ, $\nabla\psi\cdot\hat{\mathbf{n}}\,dl$ as a Fourier series on $\partial\mathcal{B}$:

$$\boldsymbol{\nabla}\psi\cdot\hat{\mathbf{n}}\,dl \equiv \frac{\partial\psi}{\partial\eta} = \sum_{k=-\infty}^{\infty} a_k \exp(2\pi ikl/L).$$

The Green's function can be similarly Fourier analyzed since both $\mathbf{r}$ and $\mathbf{r}_0$ are needed only along the boundary in Eq. (7):

$$\overline{G}(\mathbf{r},\mathbf{r}_0) = \sum_{n,n_0=-\infty}^{\infty} \overline{G}_{nn_0}\exp[2\pi i(n_0 l_0 - nl)/L].$$

Inserting these last two expressions into Eq. (7) and doing the integral over l yields the equation

$$\sum_{n_0=-\infty}^{\infty}\exp(2\pi i n_0 l_0/L)\left[\sum_{n=-\infty}^{\infty}\overline{G}_{nn_0}a_n\right] = 0,$$

whose nontrivial solutions exist only if

$$\det|\overline{G}_{nn_0}| = 0, \tag{8}$$

which is a quantization condition.

III. GREEN'S FUNCTION

To apply the method developed in the preceding section, we have to obtain the Green's function for the problem whose boundary condition is not that of the billiard. Following the procedure of the field-free case, we choose to work with the open problem, without boundaries. In the case of a constant and uniform magnetic field applied in the direction perpendicular to the plane of the billiard, this Green's function can be easily obtained. The Hamiltonian is given by $(q=m=c=1)$

$$H = \frac{1}{2}(\mathbf{p}-\mathbf{A})^2. \tag{9}$$

Choosing the symmetric gauge $\mathbf{A} = (Br/2)\hat{\boldsymbol{\theta}}$, we may write H in polar coordinates as

$$H = -\frac{\hbar^2}{2r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) - \frac{\hbar^2}{2r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\omega^2 r^2}{2} + i\hbar\omega\frac{\partial}{\partial\theta}, \tag{10}$$

where $\omega = B/2$ is half of the cyclotronic frequency.

The Green's function satisfies the equation

$$(E-H)G(\mathbf{r},\mathbf{r}_0;E) = -4\pi\delta(\mathbf{r}-\mathbf{r}_0). \tag{11}$$

This equation, however, does not specify G completely. In the case of zero magnetic field, one imposes that G represents an outgoing wave whose amplitude goes to zero as $|\mathbf{r}-\mathbf{r}_0|$ goes to infinity. In the present case such conditions cannot be imposed on G, as we shall see below. We impose instead that G has to be compatible with the corresponding Green's function for zero magnetic field [18,4]:

$$\lim_{|B|\to 0} G(\mathbf{r},\mathbf{r}_0;E) = \frac{2i\pi}{\hbar^2}H_0^{(1)}\left(\frac{\sqrt{2E}}{\hbar}|\mathbf{r}-\mathbf{r}_0|\right), \tag{12}$$

where $H_0^{(1)}$ is the zeroth-order Hankel function of the first kind.

The Green's function singularity in two dimensions is logarithmic, independent of the magnetic field, as one can easily derive by noting that it has no angle dependence for $\mathbf{r}\to\mathbf{r}_0$. Integrating (11) in a circle around $\mathbf{r}=\mathbf{r}_0$, we obtain

$$\lim_{\mathbf{r}\to\mathbf{r}_0} G(\mathbf{r},\mathbf{r}_0;E) = -\frac{4}{\hbar^2}\ln|\mathbf{r}-\mathbf{r}_0|. \tag{13}$$

Conditions (12) and (13) are sufficient to determine uniquely the solution of (11).

To solve (11), we may consider $\mathbf{r}\neq\mathbf{r}_0$, imposing afterward condition (13). Introducing the relative coordinates

$$\mathbf{R}=\mathbf{r}-\mathbf{r}_0, \quad \mathbf{R}=(X,Y), \tag{14}$$

we see that the momenta are unchanged: $P_x=p_x$, $P_y=p_y$, while the Hamiltonian changes to

$$H(P_x,X,P_y,Y)=\frac{1}{2}\left[P_x-\frac{B}{2}(Y+y_0)\right]^2 +\frac{1}{2}\left[P_y+\frac{B}{2}(X+x_0)\right]^2,$$

where $(x_0,y_0)=\mathbf{r}_0$. Applying the gauge transformation

$$\mathbf{A}'(X,Y)=\mathbf{A}(X,Y)-\frac{B}{2}(y_0,-x_0,0) = \mathbf{A}-\boldsymbol{\nabla}\varphi, \tag{15}$$

where

$$\varphi(X,Y)=\frac{B}{2}(Xy_0-Yx_0), \tag{16}$$

the Hamiltonian may be written as

$$H(P_x,X,P_y,Y)=\frac{1}{2}(\mathbf{P}-\mathbf{A}')^2.$$

To obtain $G(\mathbf{r},\mathbf{r}_0;E)$ we can therefore solve (11) for $\mathbf{r}_0=\mathbf{0}$ first and then make a gauge transform on the solution

$$G(\mathbf{r},\mathbf{r}_0;E)=G(\mathbf{r}-\mathbf{r}_0,\mathbf{0};E)e^{i\varphi(\mathbf{r},\mathbf{r}_0)/\hbar}. \tag{17}$$

We now make the *ad hoc* assumption that the whole dependence of G on the angle θ is in the phase

$$G(\mathbf{r}-\mathbf{r}_0,\mathbf{0};E)=G(|\mathbf{r}-\mathbf{r}_0|,0;E). \tag{18}$$

Thus the problem is reduced to a symmetric harmonic oscillator in two dimensions, with zero angular momentum, whose solution may be readily written in terms of the confluent hypergeometric (Kummer) function [17] (see the Appendix)

$$G(R,0;E)=\frac{2}{\hbar^2}\Gamma\left(\frac{1}{2}+\frac{Eb^2}{2\hbar^2}\right)e^{R^2/2b^2}\Psi\left(\frac{1}{2}+\frac{Eb^2}{2\hbar^2},1;-\frac{R^2}{b^2}\right), \tag{19}$$

where

$$b=\sqrt{\hbar/|\omega|} \tag{20}$$

and Γ is the Euler gamma function [19].

From [20], Eq. 6.9.(17) (p. 266) provides the limit of zero magnetic field (13), while the divergence for $R\to 0$ is obtained from Eq. 6.7.(7) (p. 259), and Eq. 6.8.(5) (p. 262) of the same reference. Combining Eqs. (19) and (17), we can write the Green's function for the Hamiltonian (9) in two dimensions

$$G(\mathbf{r},\mathbf{r}_0;E) = \frac{2}{\hbar^2}\Gamma\left(\frac{1}{2}+\frac{Eb^2}{2\hbar^2}\right)\exp\left[\frac{|\mathbf{r}-\mathbf{r}_0|^2}{2b^2}+\frac{i}{\hbar}Brr_0\sin(\theta-\theta_0)\right] \times\Psi\left(\frac{1}{2}+\frac{Eb^2}{2\hbar^2},1;-\frac{|\mathbf{r}-\mathbf{r}_0|^2}{b^2}\right). \tag{21}$$

Fourier analysis

We now present a derivation of this solution based on the decomposition of the δ function in (11) in a Fourier series. The interest is twofold, first to give a direct demonstration of the *ad hoc* hypothesis (18) and second to obtain a sum rule for the Kummer function.

Writing the solution of (11) as

$$G(\mathbf{r},\mathbf{r}_0;E)=\sum_{m=-\infty}^{\infty}\frac{1}{2\pi}G_m(r,r_0;E)e^{im(\theta-\theta_0)},$$

a term by term comparison with the δ-function Fourier expansion furnishes

$$\left[E+\frac{\hbar^2}{2}\frac{d^2}{dr^2}+\frac{\hbar^2}{2r}\frac{d}{dr}-\frac{m^2\hbar^2}{2r^2}+m\hbar\omega-\frac{\omega^2r^2}{2}\right]G_m = -\frac{4\pi}{\sqrt{rr_0}}\delta(r-r_0). \tag{22}$$

Again, for $r\neq r_0$, the solutions G_m are given in terms of the Kummer functions (see the Appendix). The requirement that G_m remains finite at the origin and the discontinuity of its derivative imply that

$$G_m(r,r_0;E)=\begin{cases} A_m(r_0)\Phi\left(\alpha_m,|m|+1;-\dfrac{r^2}{b^2}\right)\left(\dfrac{r}{b}\right)^{|m|}e^{r^2/2b^2} & \text{if}\quad r<r_0 \\ B_m(r_0)\Psi\left(\alpha_m,|m|+1;-\dfrac{r^2}{b^2}\right)\left(\dfrac{r}{b}\right)^{|m|}e^{r^2/2b^2} & \text{if}\quad r>r_0, \end{cases} \tag{23}$$

where

$$\alpha_m=\frac{Eb^2}{2\hbar^2}+\frac{|m|+m\ \mathrm{sgn}(\omega)+1}{2},$$

with $\mathrm{sgn}(x)$ the signal of x.

From the symmetry of the left-hand side of (22) with respect to the exchange of r and r_0 it follows that

$$A_m(r_0)=c_m\Psi\left(\alpha_m,|m|+1;-\frac{r_0^2}{b^2}\right)\left(\frac{r_0}{b}\right)^{|m|}e^{r_0^2/2b^2},$$

$$B_m(r_0)=c_m\Phi\left(\alpha_m,|m|+1;-\frac{r_0^2}{b^2}\right)\left(\frac{r_0}{b}\right)^{|m|}e^{r_0^2/2b^2},$$

where c_m is a constant determined by the discontinuity of the Green's function at $r=r_0$. Integrating (22) from $r_0-\epsilon$ to $r_0+\epsilon$ and letting $\epsilon\to 0$, we obtain c_m in terms of the Wronskian of $\Psi(\alpha_m,|m|+1;-r^2/b^2)$ and $\Phi(\alpha_m,|m|+1;-r^2/b^2)$ [20]:

$$c_m=(-1)^m\frac{4\pi\Gamma(\alpha_m)}{\hbar^2\Gamma(|m|+1)}.$$

Therefore, the final form of the Green's function in a Fourier series is given by

$$\begin{aligned}G(\mathbf{r},\mathbf{r}_0;E)=&\frac{2}{\hbar^2}\sum_{m=-\infty}^{\infty}(-1)^m\frac{\Gamma(\alpha_m)}{\Gamma(|m|+1)}\\&\times\Psi\left(\alpha_m,|m|+1;-\frac{r_>^2}{b^2}\right)\\&\times\Phi\left(\alpha_m,|m|+1;-\frac{r_<^2}{b^2}\right)\\&\times\left(\frac{rr_0}{b^2}\right)^{|m|}e^{(r^2+r_0^2/2b^2)+im(\theta-\theta_0)}.\end{aligned}\tag{24}$$

It is easy to see from Eq. (24) that $G(\mathbf{r},\mathbf{0};E)$ depends only on $r=|\mathbf{r}|$, proving our hypothesis in (18). Moreover, it can be equated to (21) to give an addition theorem for the Kummer function $\Psi(c-a,c;-x)$ for $c=1$.

IV. APPLICATIONS

In this section, we develop an alternative quantization condition to Eq. (8) by Fourier expanding the relevant functions directly in terms of θ instead of doing it in terms of the boundary coordinate l. This expression turns out to be more useful than Eq. (8) since the Fourier expansion of the Green's function is known in the angular variable θ. A numerical application for the elliptical billiard, which is nonintegrable for nonzero magnetic field, is then presented.

A. Quantization condition

If the points on the billiard boundary can be written in polar coordinates as $(R(\theta),\theta)$ for a differentiable function $R(\theta)$, then the integral in Eq. (7) can be performed in the following way. Define the "Fourier coefficients" $F^{(1)}_{m,j}[R]$ and $F^{(2)}_{m,j}[R]$ by

$$F^{(1)}_{m,j}[R]=\frac{1}{2\pi}\int_0^{2\pi}\Phi\left(\alpha_m,|m|+1,-\frac{R^2}{b^2}\right)e^{R^2/b^2}\left(\frac{R}{b}\right)^{|m|}\times e^{i(m+j)\theta}d\theta,$$

$$F^{(2)}_{m,j}[R]=\frac{1}{2\pi}\int_0^{2\pi}\Psi\left(\alpha_m,|m|+1,-\frac{R^2}{b^2}\right)e^{R^2/b^2}\left(\frac{R}{b}\right)^{|m|}\times e^{i(m+j)\theta}d\theta$$

such that

$$\Phi\left(\alpha_m,|m|+1,-\frac{R^2}{b^2}\right)e^{R^2/b^2}\left(\frac{R}{b}\right)^{|m|}=\sum_j F^{(1)}_{m,j}[R]e^{-i(m+j)\theta},$$

$$\Psi\left(\alpha_m,|m|+1,-\frac{R^2}{b^2}\right)e^{R^2/b^2}\left(\frac{R}{b}\right)^{|m|}=\sum_j F^{(2)}_{m,j}[R]e^{-i(m+j)\theta}.$$

At the boundary, we may also expand

$$\nabla\psi\cdot\mathbf{n}\sqrt{R^2+\left(\frac{dR}{d\theta}\right)^2}=\sum_l B_l e^{-il\theta}.$$

Using the Green's function Eq. (24), the boundary integral yields

$$\sum_k\left\{\sum_j B_j\left[\sum_m\frac{\Gamma(\alpha_m)}{\Gamma(|m|+1)}(-1)^m F^{(1)}_{m,k}[R]F^{(2)}_{m,j}[R]^*\right]\right\}\times e^{-ik\theta}=0,$$

where we have defined $F^{(2)}_{m,j}[R]^*=F^{(2)}_{m,-j-2m}[R]$. By imposing that the wave functions do not vanish identically inside the billiard, the above equation always has a solution, provided the $F^{(1)}$ coefficients obey condition

$$\det\{F^{(1)}_{m,k}[R]\}=0.\tag{25}$$

Equation (8) is only a special case of this more general condition.

We can derive a similar quantization condition for the case of two boundaries: an external boundary $\partial\mathcal{B}_2=((R_2(\theta),\theta)$ and an internal one $\partial\mathcal{B}_1=(R_1(\theta),\theta)$. Assuming that the origin does not lie inside the billiard, the functions Ψ must appear explicitly in the quantization condition. Following the same steps of the above reasoning, we use the expansion

$$\nabla\psi\cdot\hat{\mathbf{n}}\sqrt{R_n^2+\left(\frac{dR_n}{d\theta}\right)^2}=\sum_l B_{ln}e^{-il\theta},$$

where $R_n=R_1,R_2$ and the index n on B_{ln} is to denote the dependence of B_l on $R_n(\theta)$. The final result, after performing the angular integration and imposing a nonvanishing condition on the wave functions inside the billiard, is a set of linear equations in the B_{ln} variables that has a nontrivial solution if the following condition holds:

$$\det\begin{Bmatrix} F^{(1)}_{m,j}[R_1] & F^{(1)}_{m,j}[R_2] \\ F^{(2)}_{m,j}[R_1] & F^{(2)}_{m,j}[R_2] \end{Bmatrix} = 0. \tag{26}$$

It is easy to see that in the case of a circular boundary Eq. (25) gives the correct energy levels, which are directly related to the zeros of Φ (see Ref. [15]). The concentric ring billiard (the circular billiard with a concentric hole inside) also yields a simple equation for the energy levels, namely,

$$E_{mn} = \frac{2\hbar}{b^2}\left[\alpha_{mn} - \frac{|m|+m+1}{2}\right], \tag{27}$$

with $\alpha_{m,n}$ being the nth solution of

$$\Phi\left(\alpha_m, |m|+1, -\frac{1}{b^2}\right) - \frac{\Phi\left(\alpha_m, |m|+1, -\frac{R^2}{b^2}\right)}{\Psi\left(\alpha_m, |m|+1, -\frac{R^2}{b^2}\right)} \times \Psi\left(\alpha_m, |m|+1, -\frac{1}{b^2}\right) = 0. \tag{28}$$

B. Numerical results for the elliptical billiard

For a numerical test of our method we have calculated the eigenvalues of the elliptical billiard for two values of the eccentricity ϵ. The ellipse major and minor axis are $(1-\epsilon^2)^{-1/4}$ and $(1-\epsilon^2)^{1/4}$, respectively (so that the billiard area is π), and we have fixed $\hbar = m = q = 1$.

Thanks to the symmetry properties of the ellipse, the complex matrix $F^{(1)}$ can be rewritten as a real matrix. In fact, it is easy to see that any billiard whose boundary has a reflection symmetry plus and inversion symmetry allows for a real $F^{(1)}$ if the integration limits are chosen appropriately. Moreover, it turns out to be numerically convenient to renormalize the hypergeometric functions Φ and Ψ in order to keep the determinant bounded when the condition (25) is imposed.

We have computed matrices $F^{(1)}$ of size 61, which have guaranteed a precision of at least 8 digits in the eigenvalues up to an energy around 150 and magnetic field B around 30. To check the numerical precision of the energy levels we have compared the results using different matrix sizes. At $B=30$, for instance, a comparison between matrix dimensions of 41 and 61 reveals that the first 40 eigenvalues agree with at least 8 digits.

Next, in Table I we compare the results at $B=25$ between the present method and that developed in Ref. [15]. We remark that the latter procedure failed to compute the eigenvalues at magnetic fields larger than 25, although it agreed very well with the results of this paper for low fields. As the field increases, however, the calculations with the method of Ref. [15] become unstable and lose precision, as can be seen from Table I. The present approach not only works better for larger fields but also allows for the calculation of a much larger number of eigenvalues. We have computed more than 200 levels at several field values with very good acuracy.

As a final comment we remark that we have not considered here billiards with corners. In this case the method has to be modified in a way similar to the ideas presented in Ref. [2]. It is not clear by now whether these modifications can be easily performed or how the outcome will compare with the simple procedure of Ref. [15]. We plan to discuss these questions in a future work.

TABLE I. First 20 eigenvalues of the elliptical billiard for $\epsilon=0.8$ and $B=25.0$ as computed with the present method (column 2) and with the method of Ref. [15] (column 3). The fourth column shows the relative error.

n	E_{n*}	E_n	Error
1	12.50384	12.50358	2.08×10^{-5}
2	12.63204	12.63155	3.88×10^{-5}
3	13.46100	13.46186	6.39×10^{-5}
4	15.97053	15.97655	3.77×10^{-4}
5	21.02476	21.07952	2.60×10^{-3}
6	29.17016	29.56061	1.34×10^{-2}
7	37.76935	37.76905	7.94×10^{-6}
8	39.74313	39.74311	5.03×10^{-7}
9	40.71738	42.61795	4.67×10^{-2}
10	45.12332	45.12327	1.11×10^{-7}
11	54.55055	54.55006	8.98×10^{-7}
12	55.84953	57.55588	3.06×10^{-2}
13	62.73663	62.73655	1.28×10^{-6}
14	65.61804	65.61764	6.10×10^{-6}
15	68.08328	68.07920	5.99×10^{-5}
16	73.50343	73.50327	2.18×10^{-6}
17	74.65306	80.49373	7.82×10^{-2}
18	85.50246	85.52483	2.62×10^{-4}
19	86.65280	88.13405	1.71×10^{-2}
20	90.70241	90.70225	1.76×10^{-6}

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APPENDIX: DERIVATION OF THE CONFLUENT HYPERGEOMETRIC EQUATION

We depart from the homogeneous case of Eq. (22), $r \neq r'$, for general m. It reduces to the θ independent case for $m=0$,

$$\left[E + \frac{\hbar^2}{2}\frac{d^2}{dr^2} + \frac{\hbar^2}{2r}\frac{d}{dr} - \frac{m^2\hbar^2}{2r^2} + m\hbar\omega - \frac{\omega^2 r^2}{2}\right]G_m = 0. \tag{A1}$$

Supposing G_m is of the form

$$G_m(r) = F_m(r^2/b^2)\left(\frac{r}{b}\right)^{|m|}\exp-\left(\frac{r^2}{2b^2}\right), \tag{A2}$$

we get the following equation for F_m:

$$z\frac{d^2F_m}{dz^2}+(|m|+1-z)\frac{dF_m}{dz} - \left(\frac{-m\ \mathrm{sgn}(\omega)+|m|+1}{2}-\frac{Eb^2}{2\hbar^2}\right)F_m=0, \quad \text{(A3)}$$

where $z=r^2/b^2$. Equation (A3) is one of the forms of the confluent hypergeometric equation [cf. Eq. 6.3.(1) p. 252, of [20]]. The solutions we have chosen are of the form $e^z\Psi(c-a,c;-z)$ and $e^z\Phi(c-a,c;-z)$, where

$$c=|m|+1, \quad \text{(A4)}$$

$$a=\frac{-m\ \mathrm{sgn}(\omega)+|m|+1}{2}-\frac{Eb^2}{2\hbar^2}. \quad \text{(A5)}$$

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